

The Dirac and Gauge Yang-Mills Fields in Self-Consistent Consideration

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The quasi-classical model in a gauge theory with the Yang-Mills (YM) field is developed. On a basis of the exact solution of the Dirac equation in the $SU(N)$ gauge field, which is in the eikonal approximation, the Yang-Mills (YM) equations containing the external fermion current are solved. The derived solutions are quantized in the quasi-classical approach. The developed model proves to have the self-consistent solutions of the Dirac and Yang-Mills equations at $N \geq 3$. Thereat the solutions take place provided that the fermion and gauge fields exist simultaneously, so that the fermion current completely compensates the current generated by the gauge field due to its self-interaction. The obtained solution are considered in the context of QCD.

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I. INTRODUCTION

A study of non-Abelian gauge fields plays an important role in the modern field theory[1–3]. The non-Abelian gauge field are a basis of QCD[4]. The knowledge of solutions of the YM equations enable us to understand features of processes in the strong interacting matter generated in collisions of heavy ions of high energies[5]. Primarily, this concerns studying the observable states of such matter as well as the processes accompanying evolution of the medium.

Studying the non-Abelian gauge fields has a very long history which started by the classic paper by C.N.Yang and R.L.Mills[6]. Since the paper[6] has issued a lot of papers[7–17] have been devoted to deriving the solutions of the YM equation in various situations. The solution of the source-less YM equation in terms of plane waves was derived in[7, 8]. A wide class of solutions of the YM equation concerns (1+3) Minkowski space-time in the presence of external sources[9–13]. The YM equation were solved[14] for the $SU(2)$ gauge field. The spherical symmetric solutions are found for the $SU(2)$ fields in some specific case of (1+2) space-time in Ref.[15]. The Dirac equation in the presence of the $SU(3)$ YM field is considered[16, 17] in terms of the confinement problem. The quark confinement in the curve space-time is studied in Ref.[18]. Rather detailed review of the paper devoting to quantizing the YM field is in the monographs by A.Slavnov and L.Faddev[19].

The consistent consideration of the strong interacting particles (generated, for example, in collisions of high energy ions), generally, demands solving the Dirac and Yang-Mills equations simultaneously. The first step in studying such problem, naturally (see Ref.[20, 21]), is an attempt to derive the solution of these equations when the YM field has the form of some modified plane wave so that both the Dirac and Yang-Mills fields will be in the confined region of space. The knowledge of the self-consistent solution of the Yang-Mills and Dirac equations in such approximation allows us to obtain the exact Green's function of a fermion field. As a result, it enables to drive both the renormalized vertex functions and effective mass of a fermion as well as to calculate the observable characteristics of the strong interacting matter generated in collisions of high energy ions[5] beyond the perturbation theory.

In the present paper the quasi-classical model in the $SU(N)$ gauge theory with the Yang Mills field is developed. The self-consistent solutions of both the nonhomogeneous Yang-Mills equation and Dirac equations in an external field are derived when the gauge Yang-Mills field is in the eikonal form. The obtained solutions are quantized in the quasi-classical approximation. It is shown that the self-consistent solutions of such equations take place when $N \geq 3$. They occur provided that the fermion and gauge fields simultaneously exist, so that a fermion current completely compensates the current generated by the gauge field due to its self-interaction. Thereat, there is no energy flux from the range of space where the fields are localized.

In the context of the multi particle problem in matter, the derived solutions mean that the Yang-Mills field is the modified circularly-polarized wave which intensity depends strongly on fermion density. In this way, there are both the individual and collective states of fermi-particles in matter. When the matter is in equilibrium the type of the fermion states depends strongly on such parameters as the temperature and density of the medium. The fermion states appear to be one-particle at rather large temperature. With decreasing the matter temperature they are rearranged so that the collective states of the fermions interacting with the YM field arise. Thereat, interaction between the

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fermions and YM field leads to the re-normalization of a fermion mass in the mean. The re-normalized mass depends significantly on the temperature of the matter.

The problem of the hadronization of an equilibrium quark-gluon plasma is considered. In the case of the hadronization into the lightest hadrons due to the phase transition of the first kind a mass of the hadron is calculated. It appears to be of the order of the mass of a free pion.

The paper is organized as follows. Sections II consists of the statement of the problem and the description of the main approaches. In Section III the Yang-Mills equation is studied when the gauge field has the form of the eikonal. Section IV is devoted to deriving the exact solution of the Dirac equation in the external $SU(N)$ gauge field being in the form of the eikonal wave. The quantum consideration of the problem is developed in Section V. The specifics of the Yang-Mills and fermion fields obtained in the framework of the developed model are studied in Section VI. In the section VII the derived results are considered in the context of QCD. The conclusion is the last section. Appendix A contains the calculation of the operator exponent appearing in solving the Dirac equation in an external non-Abelian field. The detailed calculations of some important integrals are in Appendix B.

II. THE YM EQUATIONS IN THE PRESENCE OF EXTERNAL CURRENT

We consider the $SU(N)$ gauge field A_a^ν generated by a fermion current. It satisfies the following equations[19, 22]:

$$\partial_\mu F_a^{\nu\mu}(x) - g \cdot f_{ab}^c A_\mu^b(x) F_c^{\nu\mu}(x) = -g J_a^\nu(x) \quad (1)$$

$$F_a^{\nu\mu}(x) = \partial^\nu A_a^\mu(x) - \partial^\mu A_a^\nu(x) - g \cdot f_a^{bc} A_b^\nu(x) A_c^\mu(x), \quad (2)$$

$$J_a^\nu(x) = \bar{\Psi}(x) \gamma^\nu T_a \Psi(x), \quad (3)$$

where the fermion fields $\Psi(x), \bar{\Psi}(x)$ are governed by the Dirac equation:

$$\{i\gamma^\mu (\partial_\mu + ig \cdot A_\mu^a(x) T_a) - m\} \Psi(x) = 0 \quad (4)$$

$$\bar{\Psi}(x) \left\{ i\gamma^\mu \left(\overleftarrow{\partial}_\mu - ig \cdot A_\mu^{*a}(x) T_a \right) + m \right\} = 0; \quad T_a = \frac{1}{2} \lambda_a. \quad (5)$$

Here, m is a fermion mass, g is the coupling constant; γ^ν are the Dirac matrixes, $x \equiv x^\mu = (x^0; \vec{x})$ is a vector in the Minkowski space-time; $\partial_\mu = (\partial/\partial t; \nabla)$; the Roman letters numerate a basis in the space of the associated representation of the $SU(N)$ group, so that $a, b, c = 1 \dots N^2 - 1$. We use the signature $diag(\mathcal{G}^{\mu\nu}) = (1; -1; -1; -1)$ for the metric tensor $\mathcal{G}^{\mu\nu}$. The line and "dagger" over Ψ mean the Dirac and hermitian conjugation, respectively[23]. Summing over any pair of the repeated indexes is implied.

The symbols T_a in Eqs.(3)-(5) are the generators of the $SU(N)$ group which satisfy the commutative relations and normalization condition:

$$[T_a, T_b]_- = T_a T_b - T_b T_a = i f_{ab}^c T_c; \quad f_{ab}^c = -2 i \text{Tr} ([T_a, T_b]_- T_c) \quad (6)$$

$$\text{Tr} (T_a T_b) = \frac{1}{2} \delta_{ab}; \quad (7)$$

where f_{ab}^c are the structure constant of the $SU(N)$ group, which are real and anti-symmetrical with respect to the transposition in any pair of indexes; δ_{ab} is the Kroneker symbol. In the matrix representation the operators $(2 T_a)$ coincide with the Pauli and Gell-Mann matrixes when N is equal to 2 or 3, respectively.

It directly follows from Eqs.(6), (7) that

$$[T_a, T_b]_+ = T_a T_b + T_b T_a = \frac{1}{N} \delta_{ab} + d_{abc} T_c, \quad d_{abc} = 2 \text{Tr} ([T_a, T_b]_+ T_c) \quad (8)$$

where d_{abc} is real and symmetrical with respect to the transposition in any pair of indexes.

The main goal is to derive the self-consistent solutions of of Eqs.(1)-(5) which will be localized in the confined region of space. We find the solution when the field $A_\nu^a(x)$ is in the form:

$$A_a^\nu(x) = A_a^\nu(\varphi(x)), \quad (9)$$

where $\varphi(x)$ is some scalar function in the Minkowski space-time which is such that:

$$(\partial_\mu \varphi)(\partial^\mu \varphi) \equiv k_\mu k^\mu = 0; \quad (10)$$

The last formula determines the well known eikonal approximation where $\varphi(x)$ can be interpretable as the function governing the wave surface of the field A_a^ν .

We take the axial gauge for the field $A_\mu^a(x)$:

$$\partial^\mu A_\mu^a = 0; \quad k^\mu \dot{A}_\mu^a = 0, \quad (11)$$

where the dot over the letter means differentiation with respect to the introduced variable φ .

Taking into account of both the dependence of $A_\nu^a(x)$ on the variable x via the function $\varphi(x)$ and formulae (10), (11), we derive from Eqs.(1), (2):

$$2gk^\mu f_{ab}{}^c A_\mu^b(\varphi) \dot{A}_c^\nu(\varphi) - (\partial_\mu \partial^\mu \varphi(x)) \cdot \dot{A}_a^\nu - gk^\nu f_{ab}{}^c A_\mu^b(\varphi) \dot{A}_c^\mu(\varphi) + g^2 f_{ab}{}^c f_c{}^{sr} \{A_\mu^b(\varphi) A_s^\nu(\varphi) A_r^\mu(\varphi)\} = -g J_a^\nu(x);$$

$$J_a^\nu(x) = \bar{\Psi}(x) \gamma^\nu T_a \Psi(x). \quad (12)$$

It follows from Eq.(12) that in order to derive the solution of the YM equation it is necessary to calculate the fermion current $J_a^\nu(x)$, which, in its turn, is governed by the solutions of the Dirac equation in the external field $A_a^\nu(x)$.

To do it we assume that the field $A_a^\nu(\varphi)$ can expanded as follows in the local frame:

$$A_a^\nu(\varphi) = A \left(e_{(1)}^\nu(\varphi) \cos(\varphi(x) + \varphi_a) + e_{(2)}^\nu(\varphi) \sin(\varphi(x) + \varphi_a) \right) + \mathcal{B}_a \partial^\nu \varphi(x)$$

$$e_{(1)}^\nu e_{(2)\nu} = e_{(1)}^\nu k_\nu = e_{(2)}^\nu k_\nu = 0; \quad \dot{e}_{(1)}^\nu = e_{(2)}^\nu; \quad \dot{e}_{(2)}^\nu = -e_{(1)}^\nu; \quad k^\nu \equiv \partial^\nu \varphi(x), \quad (13)$$

where $e_{(1),(2)}^\nu(\varphi)$ are the space-like 4-vectors on the wave surface $\varphi(x)$ which are independent on the group variable a ; the symbols A , \mathcal{B}_a and φ_a are some constants in the Minkowski space-time. They are determined via the initial condition of the studied problem. It is obvious that the function $\varphi(x)$ can be taken so that the field $A_\nu^a(x)$ will be localized in the confined region of space.

III. FERMIONS IN THE EXTERNAL YM FIELD

To obtain the fermion field $\Psi(x)$ we go from Eq.(4) to the so-called quadric Dirac equation which has the following form:

$$\left\{ -\partial_\mu \partial^\mu - m^2 + g^2 (\gamma^\mu A_\mu^a T_a)^2 + 2ig (\gamma^\nu A_\nu^a T_a) (\gamma^\mu \partial_\mu) + ig (\gamma^\nu k_\nu) (\gamma^\mu \dot{A}_\mu^a T_a) \right\} \Phi(x) = 0;$$

$$\Psi(x) = \left\{ \frac{i\gamma^\mu (\partial_\mu - ig \cdot A_\mu^a(x) T_a) + m}{2m} \right\} \Phi(x) \quad (14)$$

First, to derive the solution of the last equation we simplify the third term in the left-hand side of Eq.(14).

Let the initial conditions be so that the phases φ_a in Eq.(13) satisfy the equations:

$$d_a{}^{bc} \cos(\varphi_a - \varphi_b) = 0. \quad (15)$$

Then, using Eqs.(6)-(8) and relations for the γ -matrixes[23, 24] we obtain after direct calculations:

$$\begin{aligned}
(\gamma^\mu A_\mu^a T_a)^2 &= \left(\frac{1}{2N} \delta_{ab} + \frac{1}{2} d_{abc} T^c + \frac{i}{2} f_{abc} T^c \right) (\mathcal{G}^{\mu\nu} + \sigma^{\mu\nu}) A_\mu^a A_\nu^b = \frac{1}{2N} A_\mu^a A_a^\mu ; \\
\sigma^{\mu\nu} &= \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)
\end{aligned} \tag{16}$$

We note, that the term containing $(f_{abc} T^c \sigma^{\mu\nu} A_\mu^a A_\nu^b)$ disappears since the vectors $e_{(1),(2)}^\nu$, which are in the expansion (13), are independent on the group variable a . The same takes place with respect to all terms containing $\sigma^{\mu\nu}$ in Eq.(15).

Let us find the solution of Eq.(14) in the following form:

$$\Phi(x) \equiv \Phi_{\sigma,\alpha}(x, p) = e^{-ipx} \cdot F_{\sigma,\alpha}(\varphi). \tag{17}$$

where $F_{\sigma,\alpha}(\varphi)$ is some multicomponent function which is the generalized Dirac spinor. It depends on both the spin variable σ and the variable α which specifies the state of a fermion in the space of the fundamental representation of the $SU(N)$ group, thereat $\alpha = 1 \div N$; $p^\nu = (p^0, \vec{p})$ is some 4-vector.

We substitute $\Phi_{a,\alpha}(x, p)$ given by Eq.(17) into the formula (14). Using relations for the γ - matrixes [23, 24], independence of $e_{(1),(2)}^\nu$ on the group variable a in the local frame (see Eq.(13)) as well as Eq.(10), (11), (16), we obtain:

$$\begin{aligned}
\left\{ p^2 - m^2 - \frac{g^2(N^2 - 1)A^2}{2N} - 2g(T_a A_\mu^a p^\mu) - ig(\gamma^\nu k_\nu) \left(\gamma^\mu T_a \dot{A}_\mu^a \right) \right\} F_{\sigma,\alpha}(\varphi) + i(pk) \dot{F}_{\sigma,\alpha}(\varphi) &= 0; \\
k_\mu &= \partial_\mu \varphi(x); \quad (pk) = p^\mu k_\mu;
\end{aligned} \tag{18}$$

where the dot over $\dot{F}_{\sigma,\alpha}(\varphi)$ means derivative with respect to the variable φ .

In obtaining the last equation we neglect $|\partial_\mu k^\mu|$ as compared with $|(pk)|$ (see Eq.(18)). This means that the wave length λ_{YM} of the YM field is unchangeable on the scale which is of the order of the de Broglie wave length of a fermion λ_F :

$$|\partial_\nu k^\mu| \lesssim |\partial_\mu k^\mu| = |\partial_\mu \partial^\mu \varphi(x)| \sim \left| \frac{d\lambda_{YM}}{\lambda_{YM}^2 dx} \right| \ll \frac{1}{\lambda_{YM} \lambda_F} \sim |(pk)| \Leftrightarrow \left| \frac{d\lambda_{YM}}{dx} \right| \ll \frac{\lambda_{YM}}{\lambda_F} \ll 1 \tag{19}$$

The last inequality corresponds to the so called quasi-classical approximation and means that $\partial_\mu k^\mu = 0$. The condition $\partial_\mu k^\mu = 0$ can be treated as the scale invariance of the wave surface of the YM field. Thereat, the form of the wave surface is determined by the harmonic functions satisfying D'Lambert equation, $\partial_\mu \partial^\mu \varphi(x) = 0$.

In the quasi-classical approximation governed by the inequality (19) we neglect the unexplicit dependence of k^μ on the eikonal $\varphi(x)$ and assume that k^μ only varies along the wave surface. Then, the solution of Eq.(18) can be written in the form:

$$\begin{aligned}
F_{\sigma,\alpha}(\varphi) &= \exp \left(-ig^2 \frac{(N^2 - 1)A^2}{2N(pk)} \right) \exp \left\{ -ig T_a \frac{\frac{i}{2} (\gamma^\nu k_\nu) (\gamma^\mu A_\mu^a(\varphi)) + \int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu)}{(pk)} \right\} u_\sigma(p) \cdot v_\alpha, \\
p^2 &= m^2; \quad \partial_\mu k^\mu = \partial_\mu \partial^\mu \varphi(x) = 0;
\end{aligned} \tag{20}$$

where $u_\sigma(p)$ and v_α are spinors which are elements in the space of the corresponding representations.

The second exponent in Eq.(20) is the operator acting on the spinors $u_\sigma(p)$ and v_α . The transforming the exponent in the same way as it has been done in Ref.[25] (see also Appendix A), we obtain:

$$\begin{aligned}
& \exp \left\{ -ig T_a \frac{\frac{i}{2} (\gamma^\nu k_\nu) (\gamma^\mu A_\mu^a(\varphi)) + \int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu)}{(pk)} \right\} = \\
& \cos \theta \left\{ \left(1 - ig T_a \frac{\tan \theta}{\theta(pk)} \int_0^\varphi d\varphi' (A_\mu^a p^\mu) \right) + \frac{g (\gamma^\nu k_\nu) (\gamma^\mu A_\mu^a)}{2(pk)} \cdot \left[\frac{\tan \theta}{\theta} T_a + \right. \right. \\
& \left. \left. \frac{g}{(pk)} \frac{1}{2N} \left(-i \frac{\tan \theta}{\theta} + \frac{g}{(pk)} \frac{\theta - \tan \theta}{\theta^3} T_b \int_0^\varphi d\varphi' (A_\mu^b p^\mu) \right) \int_0^\varphi d\varphi' (A_\nu^a p^\nu) \right] \right\}; \\
& \theta = \frac{g}{(pk)} \sqrt{\frac{1}{2N}} \left(\int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu) \int_0^\varphi d\varphi'' (A_\nu^a(\varphi'') p_\nu) \right)^{\frac{1}{2}}
\end{aligned} \tag{21}$$

We substitute the exponent given by the last formula into Eq.(20). After that, using Eqs.(17) and (14) we obtain

$$\begin{aligned}
\Psi_{\sigma,\alpha}(x,p) &= \Phi_{\sigma,\alpha}(x,p) = \cos \theta \cdot \exp \left(-ig^2 \frac{(N^2 - 1)A^2}{2N(pk)} \varphi - ipx \right) \left\{ \left(1 - ig T_a \frac{\tan \theta}{\theta(pk)} \int_0^\varphi d\varphi' (A_\mu^a p^\mu) \right) + \right. \\
& \left. \frac{g (\gamma^\nu k_\nu) (\gamma^\mu A_\mu^a)}{2(pk)} \cdot \left[\frac{\tan \theta}{\theta} T_a + \frac{g}{(pk)} \frac{1}{2N} \left(-i \frac{\tan \theta}{\theta} + \frac{g}{(pk)} \frac{\theta - \tan \theta}{\theta^3} T_b \int_0^\varphi d\varphi' (A_\mu^b p^\mu) \right) \int_0^\varphi d\varphi' (A_\nu^a p^\nu) \right] \right\} u_\sigma(p) \cdot v_\alpha; \\
\theta &= \frac{g}{(pk)} \sqrt{\frac{1}{2N}} \left(\int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu) \int_0^\varphi d\varphi'' (A_\mu^a(\varphi'') p_\mu) \right)^{\frac{1}{2}}; \quad (\partial_\nu k^\nu) = (\partial_\nu \partial^\nu) \varphi(x) = 0.
\end{aligned} \tag{22}$$

A. General Solution of Dirac Equation in External Field

In order to derive the general solution of the Dirac equation we need to specify the physical sense of the spinors $u_\sigma(p)$, v_α and the 4-vector p which are in Eq.(22).

First, we require that the wave function (22) coincides with the solution of the free Dirac equation at $\varphi = 0$. This means that the spinors $u_\sigma(p)$ satisfy the relations:

$$\sigma^{\mu\nu} k_\mu A_\nu(\varphi = 0) u_\sigma(p) = 0; \quad \bar{u}_\sigma(p) u_\lambda(p') = \pm 2m \delta_{\sigma\lambda} \delta_{pp'}; \quad p^2 = m^2, \tag{23}$$

where $u_\sigma(p)$ are the bispinors of the free Dirac field. Thereat, the first relation in Eq.(23) fixes the fundamental set of solutions of Eq.(14) which is determined by the parameter p . The plus and minus signs in Eq.(23) correspond to the Dirac scalar production of the spinors $u_\sigma(p)$ and $u_\sigma(-p)$, respectively.

Let us clarify the physical sense of the 4-vector p^μ arising in Eq.(17). To do it we consider the projections of the momentum operator, $\hat{p}^\mu = -i\partial^\mu$, along the k^μ -direction and on the plane which is perpendicular to the vector k^μ in the local frame govern by Eq.(13). For definiteness sake, we take $k^\mu = (1, 0, 0, 1)$; $e_{(1)}^\mu = (0, 1, 0, 0)$; $e_{(2)}^\mu = (0, 0, 1, 0)$.

Then, the function (22) is the eigenfunction of the operators \hat{p}^1 , \hat{p}^2 and $\hat{p}^0 - \hat{p}^3$ which eigenvalues are p^1 and p^2 and $p^0 - p^3$, respectively. Thereat, the operators \hat{p}^1 , \hat{p}^2 and $\hat{p}^0 - \hat{p}^3$ commute with the Hamiltonian \mathcal{H} :

$$\mathcal{H} = \gamma^0 \vec{\gamma} \left(\hat{\vec{p}} - g T^a \vec{A}_a \right) + \gamma^0 m + g A^0; \quad A_a^\mu = (A^0; \vec{A}), \tag{24}$$

This means that the combinations $P^1 = p^1$; $P^2 = p^2$; $P^3 = p^0 - p^3$ of the components of the vector $p^\mu = (p^0; \vec{p})$ introduced by Eq.(17) are the quantum numbers of the solutions of the Dirac equation (14).

As for the spinor v_α , we determine it by the relations:

$$v_\alpha^\dagger v_\beta = \delta_{\alpha\beta}; \quad Tr(T_a) = 0; \quad Tr(T_a T_b) = \frac{1}{2}\delta_{ab} \quad (25)$$

Then, the function (22) can be normalized by the δ -function as follows:

$$\int d^3x \Psi_{\sigma,\alpha}^*(x, p') \Psi_{\sigma,\alpha}(x, p) = (2\pi)^3 \delta^3(\vec{p} - \vec{p}'). \quad (26)$$

Direct calculations show that $\Phi_{\sigma,\alpha}(x, p)$ and $\Phi_{-\sigma,\alpha}(x, -p)$ are orthogonal. In this way, it is obvious, that $\Phi_{\sigma,\alpha}(x, p)$ is the so-called positively frequency function whereas $\Phi_{-\sigma,\alpha}(x, -p)$ is negatively frequency one[16, 17]. This fact allows us to construct the general solution of the Dirac equation which describes the states both particles and anti-particles. Since the quantum number $P^1 = p^1$; $P^2 = p^2$; $P^3 = p^0 - p^3$ are linearly related to the vector \vec{p} we can derive the general solution of the Dirac equation by combining the functions $\Phi_{\sigma,\alpha}(x, p)$ and $\Phi_{-\sigma,\alpha}(x, -p)$. As a result, due to the completeness condition (26) the general solution of Eq.(14) is

$$\begin{aligned} \Psi(x) &= \sum_{\sigma,\alpha} \int \frac{d^3p}{\sqrt{2p^0} (2\pi)^3} \left\{ \hat{a}_{\sigma,\alpha}(\vec{p}) \Psi_{\sigma,\alpha}(x, p) + \hat{b}_{\sigma,\alpha}^\dagger(\vec{p}) \Psi_{-\sigma,\alpha}(x, -p) \right\} \\ \bar{\Psi}(x) &= \sum_{\sigma,\alpha} \int \frac{d^3p}{\sqrt{2p^0} (2\pi)^3} \left\{ \hat{a}_{\sigma,\alpha}^\dagger(\vec{p}) \bar{\Psi}_{\sigma,\alpha}(x, p) + \hat{b}_{\sigma,\alpha}(\vec{p}) \bar{\Psi}_{-\sigma,\alpha}(x, -p) \right\}, \end{aligned} \quad (27)$$

where the symbols $\hat{a}_{\sigma,\alpha}^\dagger(\vec{p})$; $\hat{b}_{\sigma,\alpha}^\dagger(\vec{p})$ and $\hat{a}_{\sigma,\alpha}(\vec{p})$; $\hat{b}_{\sigma,\alpha}(\vec{p})$ are the operators of creation and cancellation of a fermion ($\hat{a}_{\sigma,\alpha}(\vec{p})$; $\hat{a}_{\sigma,\alpha}^\dagger(\vec{p})$) and anti-fermion ($\hat{b}_{\sigma,\alpha}(\vec{p})$; $\hat{b}_{\sigma,\alpha}^\dagger(\vec{p})$), respectively[22, 23]. Thereat, $\hat{a}_{\sigma,\alpha}(\vec{p})$; $\hat{a}_{\sigma,\alpha}^\dagger(\vec{p})$; $\hat{b}_{\sigma,\alpha}(\vec{p})$ and $\hat{b}_{\sigma,\alpha}^\dagger(\vec{p})$ satisfy the standard commutative relations for the fermion operators.

IV. SOLUTION OF YM EQUATION IN THE EIKONAL APPROXIMATION

Let us fix the state of the fermion vacuum so that the bilinear combinations of the operators of creation and cancellation of fermions are diagonal. Substituting $\bar{\Psi}(x)$ and $\Psi(x)$ given by Eq.(22), (27) into the formula (3) we derive the following after direct calculations:

$$\begin{aligned} J_a^\nu &= \bar{\Psi}(x) \gamma^\nu T_a \Psi(x) = \cos^2 \theta \sum_{\sigma,\alpha} \int \frac{d^3p}{p^{(0)} (2\pi)^3} \left\{ -g A_a^\nu \frac{\tan \theta}{2N\theta} - \right. \\ &g^3 \frac{\theta \tan^2 \theta + \theta - \tan \theta}{4N^2(pk)^2 \theta^3} \int_0^\varphi d\varphi' (A_a^\mu(\varphi') p_\mu) \left(A_b^\nu(\varphi) \int_0^\varphi d\varphi' (A_\mu^b(\varphi') p^\mu) \right) + \frac{gk^\nu}{(pk)} \cdot \\ &\left[\frac{\tan \theta}{2N\theta} (A_a^\mu(\varphi) p_\mu) + g^2 \frac{\theta \tan^2 \theta + \theta - \tan \theta}{4N^2(pk)^2 \theta^3} \int_0^\varphi d\varphi' (A_a^\mu(\varphi') p_\mu) \right. \\ &\left. \left((A_b^\mu(\varphi) p_\mu) \int_0^\varphi d\varphi' (A_\mu^b(\varphi') p^\mu) \right) \right] \left\{ \langle \hat{a}_{\sigma,\alpha}^\dagger(p) \hat{a}_{\sigma,\alpha}(p) + \hat{b}_{\sigma,\alpha}(p) \hat{b}_{\sigma,\alpha}^\dagger(p) \rangle \right\} \end{aligned} \quad (28)$$

where the angle brackets mean averaging over the vacuum state of fermions.

When the fermion system is homogeneous and isotropic the integrals containing the square bracket are equal to zero owing to the relativistic invariance (see Appendix B). Then, we substitute the current given by Eq.(28) and the field A_a^ν governed by Eq.(13) into Eq.(20). In taking into account of Eqs.(10) (20) (that leads to cancellation of the first two terms in the left-hand side of Eq.(12)), we derive:

$$2f_{ab}^c \sin(\varphi_b - \varphi_c) = f_{ab}^c \left\{ f_c^{sr} \cos(\varphi_b - \varphi_r) + \{\cos(\varphi_b - \varphi_r) \cos(\varphi_s - \varphi_a)\} \frac{f_c^{bs}}{N} \right\} \mathcal{B}_s; \quad (29)$$

$$A^2 \cdot C = -(N^2 - 1) \sum_{\sigma\alpha} \int \frac{d^3 p}{p^{(0)}(2\pi)^3} \langle \hat{a}_{\sigma,\alpha}^\dagger(\vec{p}) \hat{a}_{\sigma,\alpha}(\vec{p}) + \hat{b}_{\sigma,\alpha}(\vec{p}) \hat{b}_{\sigma,\alpha}^\dagger(\vec{p}) \rangle, \quad (30)$$

where

$$C = f_{ab}^c f_c^{sr} \{\cos(\varphi_b - \varphi_r) \cos(\varphi_s - \varphi_a)\} < 0. \quad (31)$$

The equations (29), (30) are closed with respect to the unknown quantities A and \mathcal{B}_a . Having been solved they determine both the fermion and gauge field by means of Eqs.(13), (22), (27) so that the wave surface $\varphi(x)$ is governed by the relations (10), (19), (20).

Note that in the case of the $N = 2$ (when the $SU(2)$ gauge symmetry occurs) the convolution (31), containing cosines, always is positive since the structure constants f_{ab}^c are the completely antisymmetrical tensor of the third rang ε_{ab}^c due to the Jacob equality[22]:

$$C = f_{ab}^c f_c^{sr} \{\cos(\varphi_b - \varphi_r) \cos(\varphi_s - \varphi_a)\} = \sum_{a,b=0}^{N^2-1} \sin^2(\varphi_s - \varphi_a) \geq 0. \quad (32)$$

This means that in the framework of the developed model there is no self-consistent solution of the Dirac and Yang-Mills equations in the case of the $SU(2)$ gauge symmetry. When the group dimension is more then $N = 2$ the structure constants f_{ab}^c can not be expressed in terms of the tensor ε_{ab}^c . As a result, it possible to fix the differences between phase in the convolution C so that $C \leq 0$.

As for the coefficients \mathcal{B}_s they satisfy the set of linear algebraical equations. The matrix of this set is symmetrical and, moreover, its diagonal elements are not all equal to zero. This means that the equation for \mathcal{B}_s has the unambiguous solution.

As a result, we have the following. The problem governed by Eqs.(1)-(5) has the unique solution when $N \geq 3$. The solutions are determined by Eqs.(13), (27), (29), (30) and correspond to the eikonal consideration when the wave surface of the fields are determined by the equations:

$$(\partial_\mu \varphi(x)) \cdot (\partial^\mu \varphi(x)) = 0; \quad (\partial_\mu \partial^\mu) \varphi(x) = 0 \quad (33)$$

It follows from Eqs. (13), (27), (29), (30) that the Yang-Mills and Dirac equations have the self-consistent solution when the fermion current compensates the current of the gauge field which takes place due to self-interaction of such field. In other words, in the the framework of the developed model there is no the YM field without fermions. In terms of QCD this means that quarks and gluons can not exist separately in such approach.

We should note here that the second relation in Eq.(33) implies that the function $\varphi(x)$ which is the argument in the expansion (13) of the field A_a^ν is the so called harmonic function. Owing to the initial conditions it can be always taken such that the field A_a^ν will be localized in the confined region of space. In this way, the relations (10), (33) and (13) directly lead to the axial gauge (11).

V. QUANTIZING THE YANG-MILLS FIELD

We note, that the formula (13) determining the YM field can be rewritten as follows:

$$\begin{aligned} A_a^\nu(x) &= \frac{1}{\sqrt{V}\sqrt{2\omega}} \sum_{\vec{q}, \alpha} \{c_\alpha(\vec{q}) e^\nu \exp(-i\varphi_a) \exp(-iqx) + c_\alpha^*(\vec{q}) e^{*\nu} \exp(i\varphi_a) \exp(iqx)\} + \mathcal{B}_a k^\nu \\ c_\alpha(\vec{q}) &= A\sqrt{V}\sqrt{2\omega} \int \frac{d^3 \vec{x}}{2(2\pi)^3} \exp(i \vec{q} \cdot \vec{x} - i\varphi(x)); \\ e^\nu e_\nu &= e^{*\nu} e_\nu^* = 0; \quad e^\nu e_\nu^* = 1; \quad q = (\omega; \vec{q}); \quad q^\mu q_\mu = 0, \end{aligned} \quad (34)$$

where $\alpha = 1 \div 2$ takes into account of two polarizations of the YM field, V is the normalizing volume.

Let us change the coefficients $c_\alpha(\vec{q})$ and $c_\alpha^*(\vec{q})$ by the operators of cancellation $\hat{c}_\sigma(\vec{q})$ and creation $\hat{c}_\sigma^\dagger(\vec{q})$ of a quant of the YM field so that $\hat{c}_\sigma(\vec{q})$ and $\hat{c}_\sigma^\dagger(\vec{q})$ satisfy the Bose-Einstein commutative relation. In the quasi-classical approximation (19) this means that $\hat{c}_\sigma(\vec{q})$ and $\hat{c}_\sigma^\dagger(\vec{q})$ commute each other:

$$\hat{c}_\sigma(\vec{q}) \hat{c}_\sigma^\dagger(\vec{q}) \approx \hat{c}_\sigma^\dagger(\vec{q}) \hat{c}_\sigma(\vec{q}), \quad (35)$$

where σ is the spin variable.

Taking into account of Eqs.(10) and (20), we substitute Eq.(34) into the formula (12). As a result, we derive the relation between the occupancy number of fermions and YM quants:

$$\frac{C}{2} \sum_{\vec{q}, \sigma} \frac{1}{2\omega(\vec{q})} \{c_\sigma^\dagger(\vec{q}) c_\sigma(\vec{q})\} = -(N^2 - 1) \sum_{\sigma; \alpha; \vec{p}} \frac{1}{p^{(0)}(\vec{p})} \left\{ \hat{a}_{\sigma, \alpha}^\dagger(\vec{p}) \hat{a}_{\sigma, \alpha}(\vec{p}) + \hat{b}_{\sigma, \alpha}(\vec{p}) \hat{b}_{\sigma, \alpha}^\dagger(\vec{p}) \right\}, \quad (36)$$

where all notations are the same as they are in Eq.(30), (31), (34).

Eqs.(27), (35) allow us to obtain the energy E of interacting fermions. Calculating the energy-momentum tensor $T^{\nu\mu}$ we derive

$$E = \sum_{\vec{q}, \sigma} \omega(\vec{q}) \{c_\sigma^\dagger(\vec{q}) c_\sigma(\vec{q})\} + \sum_{\sigma; \alpha; \vec{p}} p^{(0)}(\vec{p}) \left\{ \hat{a}_{\sigma, \alpha}^\dagger(\vec{p}) \hat{a}_{\sigma, \alpha}(\vec{p}) + \hat{b}_{\sigma, \alpha}(\vec{p}) \hat{b}_{\sigma, \alpha}^\dagger(\vec{p}) \right\}. \quad (37)$$

It follows from the last expression that the energy of the system of particles which consists of fermions and gauge quants splits on two terms. However such additivity is fictitious since the occupancy numbers of the fermion field depend on the value of the YM field. Formally, this manifests itself via Eq.(36) at the quasi-classical level.

VI. DIRAC AND YANG-MILLS FIELDS AS STRONG INTERACTING MATTER

Let us consider the solution of Eq.(29)-(30) in detail, when $N \geq 3$. We assume that the phases φ_a are chosen so that the convolution C given by Eq.(31) is negative.

A. Fermion fields

In principle, Eqs. (13), (27), (30) allow us to calculate both the amplitude and phases of the YM field provided that the occupancy number of fermions is known. However, the relation (30) is some functional equation since the right-hand side of it depends on the required value A via the 4-momentum $p^\mu = (E(\vec{p}); \vec{p})$ of a fermion in an external field (see Eq.(27)) which enters into the correlators of fermi-operators. Such complicate problem is simplified and can be solved in a very important case when the system of fermions is some equilibrium matter whose temperature is T . Then, the correlators in Eq.(30) are the equilibrium occupancy numbers $n(E)$ which are equal to [26]:

$$\langle \hat{a}_{\sigma, \alpha}^\dagger(p) \hat{a}_{\sigma, \alpha}(p) \rangle = \langle \hat{b}_{\sigma, \alpha}(p) \hat{b}_{\sigma, \alpha}^\dagger(p) \rangle \cong \frac{1}{1 + \exp\left(\frac{p^\mu U_\mu - \mu}{T}\right)} \cong \frac{1}{1 + \exp\left(\frac{Q^\mu U_\mu - \mu}{T}\right)} \equiv n(Q^0), \quad (38)$$

where μ is the chemical potential supposed to be the same for all type of fermions; Q^μ is the mean value of a kinetic momentum of a fermion, $Q^\mu = p^\mu - gT^a A_a^\mu$, in an external field; $U_\mu = (1, 0, 0, 0)$ is the so-called hydrodynamics velocity[26].

The functions (22) allow us to drive the mean value of the kinetic momentum of a fermion Q^μ . After a direct calculations we get

$$(Q^0)^2 = \vec{p}^2 + m_*^2; \quad m_*^2 = m^2 + \frac{g^2(N^2 - 1)A^2}{2N} \quad (39)$$

Since Q^0 can be interpretable as the mean value of the energy E of a fermion in an external field, the last equation means that the interaction of a fermion with an external field leads to the re-normalization of a fermion mass, in the mean.

Substituting Eqs.(27), (38) into the formula (30), we obtain:

$$A^2 \cdot |C| = 2N(N^2 - 1) \left(\frac{T}{\pi} \right)^2 \int_z^\infty \frac{\sqrt{x^2 - z^2} dx}{1 + \exp\left(x - \frac{\mu}{T}\right)}, \quad z = \frac{m_*}{T} = \frac{1}{T} \sqrt{m^2 + \frac{g^2(N^2 - 1)A^2}{2N}}, \quad (40)$$

To derive the solution of Eq.(40), first, we assume that the occupancy numbers are not too large so that the Boltzmann distribution is applicable to fermions. In this case the chemical potential can be expressed via the density of particles n_0 [26]. Then, we obtain from Eq.(40) at $T \gg m$:

$$z^3 K_2(z) = \frac{n_0}{|C|T^3} K_1(z); \quad T \gg m, \quad (41)$$

where $K_\nu(z)$ are the modified Bessel functions (the McDonald functions)[27] .

The ratio $K_1(z)/K_2(z)$ monotonically increases, so that[27]:

$$\begin{aligned} K_1(z)/K_2(z) &= z/2; & z \rightarrow 0 \\ K_1(z)/K_2(z) &= 1; & z \rightarrow \infty. \end{aligned} \quad (42)$$

Then, Eq.(41) has the unambiguous solution which is

$$\begin{aligned} z &= \left(\frac{n_0}{2|C|T^3} \right)^{1/2}; & z \ll 1 \\ z &= \left(\frac{n_0}{|C|T^3} \right)^{1/3}; & z \gg 1 \end{aligned} \quad (43)$$

Since the Boltzmann approximation is correct when $(n_0/T^3) \ll 1$, the first formula in Eq.(43) can only be used for calculation of the amplitude A of the YM field. Then, we derive from Eq.(40):

$$A = \frac{2N}{g(N^2 - 1)} \left(\frac{n_0}{2|C|T^3} \right)^{1/2} T; \quad \frac{m}{T} \ll \left(\frac{n_0}{|C|T^3} \right) \ll 1; \quad (44)$$

It follows from Eq.(44) that at $(n_0 T^{-3}) \lesssim 1$ the field amplitude A is such that the effective mass of a fermion is small, $m_* \ll T$, even in the presence of the external field, i.e. the fermions in the external field remain an ultrarelativistic particles as before.

On the other hand, the wave packet of an ultrarelativistic $E \gg m$ particle does not spread out as compared with the case of non-relativistic particles owing to the dispersion law which is $E(p) \sim p$ at $E \gg m$. This means that fermion states in the external field are single-particle ones in this case. Then, the system of fermions can be considered as some matter consisting of individual fermi-particles so that the interaction of them with the field results in the renormalization of their masses, in the mean. In this way, the individuality of a particle keeps as soon as the density of the matter is not too large $(n_0 T^{-3}) \lesssim 1$ [28].

With increasing the number of fermions (or with decreasing the matter temperature) the Boltzmann approach becomes unsuitable. In the case of $n_0 \gtrsim T^3$, the chemical potential μ is of the order of $\mu \sim n_0^{2/3} \gg T$. Then, transforming the integral in Eq.(40) according to Ref.[28], we obtain:

$$A = \left(\frac{2N}{g(N^2 - 1)} \right)^{\frac{1}{2}} \mu \simeq \left(\frac{2N}{g(N^2 - 1)} \right)^{\frac{1}{2}} n_0^{\frac{2}{3}} \gg T; \quad \frac{\mu}{T} \gtrsim z \gg 1. \quad (45)$$

The last formula shows that in decreasing the temperature of matter (or in increasing its density) the mean effective mass of a fermion m_* in the external field is enlarged, so that the fermions become non-relativistic particles. That leads to delocalizing the fermion states in the space due to spreading out of a wave function. Thus, in this case the matter constitutes some fermi-liquid consisting of fermions with the renormalized mass.

B. Yang-Mills fields

The obtained formulae (29), (30) allow us to get the tensor $F_a^{\nu\mu}(x)$ of the gauge YM field. After direct calculations we get from Eqs.(2), (13):

$$F_a^{\nu\mu} = (\partial_\nu \varphi) \frac{\partial A_a^\mu}{\partial \varphi} - (\partial_\mu \varphi) \frac{\partial A_a^\nu}{\partial \varphi}. \quad (46)$$

The derived tensor $F_a^{\nu\mu}$ enables us to obtain the strength of both a "electric" and "magnetic" field as well as the energy-momentum tensor $T^{\nu\mu}$. The diagonal components of $T^{\nu\mu}$ give the energy density w and Pointing vector \vec{S} :

$$T^{\nu\nu} = (\partial^\nu \varphi) \cdot (\partial^\nu \varphi) A^2 \equiv (w; \vec{S}). \quad (47)$$

The last formulae show that there is no energy flux of the gauge field through any surface confining the range where the YM field is. This means that the role of the YM field in the considered approximation is binding fermion what leads to the renormalization of their masses.

VII. DEVELOPED MODEL IN CONTEXT OF QCD

A. Effective mass of quarks

First, we discuss applicability of the developed model to description of the strong interacting matter generated in collisions of heavy ions of high energies. The quasi-classicality of the model means that the occupancy number of particle are large.

In the RHIC and SPS experiments the characteristic temperature T of an equilibrium quark-gluon plasma is $T \sim 200 \div 400 \text{ MeV}$. The estimations of the initial density of energy of the plasma give that the energy density $w \sim 10 \text{ GeV} \cdot F^{-3}$ while the volume of the fireball is not less than $V_0 \sim 10^2 F^3$. Then the number of particles N inside the fireball is of the order of

$$N \sim \frac{w V_0}{T} \gtrsim 2.5 \cdot 10^3, \quad (48)$$

that is in agreement with the quasi-classical approximation.

The gas parameter $n_0^{1/3} T^{-1}$ is of the order of $(n_0^{1/3} T^{-1}) \sim 1.46 \div 3.7$ at such density of the matter. On the other hand, the mean effective mass of a quark is of the order of

$$\begin{aligned} m_* &\sim \left(\frac{2N}{g(N^2 - 1)} \right)^{\frac{1}{2}} \left(\frac{n_0}{2|C|T^3} \right)^{1/2} T; & \frac{m}{T} &\ll \left(\frac{n_0}{|C|T^3} \right) \ll 1; \\ m_* &\sim \sqrt[3]{n_0}; & \left(\frac{n_0}{|C|T^3} \right) &\gg 1. \end{aligned} \quad (49)$$

It follows from the last formulae that in the intermediate range of the density of matter $n_0 \sim (gT)^3$ the effective mass is proportional to the temperature of the matter that corresponds to the result of the calculation of the thermal mass of a quark in the hard loop approximation[29, 30]:

$$m_* \sim g T. \quad (50)$$

B. Hadronization

Arising the collective fermi-liquid states of fermions, which are governed by Eqs.(40), (45), is typical in the situation when there are no any channel for the particles to escape the fermion system. In the case of a quark-gluon plasma quarks can go out of the system due the process of the hadronization.

We estimate the mass of the hadron generated in the result of the hadronization. We assume that the hadronization is the equilibrium phase transition of the first kind. Then, the chemical potentials of the quarks μ_q , gluons μ_g and hadrons μ_h are:

$$\mu_q = \mu_g = \mu_h = 0 \quad (51)$$

Substituting $\mu_q = 0$ into Eq.(40) we derive:

$$A \sim 1.5 T_c; \quad m_* \sim 1.78 T_c, \quad (52)$$

where T_c is the temperature of the considered phase transition. In obtaining Eqs.(52), we set $g = 1$; $|C| \simeq 1$.

Let us the pions π^0 are only created in the result of the phase transition. Since π^0 's consist of 12 quarks (including anti-quarks) we equalize the number of hadrons and quarks (divided by 12) at the temperature T_c . As a result, we obtain:

$$\frac{3}{2} \int_0^\infty \frac{x^2 dx}{\exp\left(\sqrt{x^2 + (M_h/T_c)^2}\right) - 1} = \int_0^\infty \frac{x^2 dx}{\exp\left(\sqrt{x^2 + (1.78)^2}\right) + 1}, \quad (53)$$

where M_h is the hadron mass. In deriving the last equation we take into account that the effective mass of a quark is $m_* \sim 1.78 T_c$ according to Eqs.(52).

Solution of Eq.(53) gives that $M_h \simeq 0.68 T_c$. If the temperature of the phase transition is $T_c \simeq 200 MeV$, the hadron mass is $M_h \simeq 136 MeV$, that corresponds to the mass of a free pion.

VIII. CONCLUSION

The quasi-classical model in the gauge $SU(N)$ field theory is considered when the YM field is assumed to be in the form of the eikonal wave. The self-consistent solutions of the non-homogeneous YM equation and the Dirac equation in the external YM field is derived. It is shown that the considered problem is solvable when the dimension of the gauge group $N \geq 3$. Thereat, the currents generated by fermions and gauge field exactly compensate each other.

In terms of the multi particle problem, the obtained solutions correspond to the both individual and collective states of fermions in matter that depends strongly on the parameters of the problem such as the density of fermions and the temperature of matter. As for the YM field, its amplitude appears to depend strongly on the number of fermions so that the field does not exist without fermions. The derived gauge field has the form of a circulatory polarized wave (see Eqs.(13)) which energy is concentrated in the localized region of space. Thereat, interaction of the YM field with fermions leads, in the mean, to the re-normalization of a fermion mass so that it enlarges with increasing the YM field amplitude.

The quantum theory of the considered model is developed in the quasi-classical approximation. The energy of the quantized fields is obtained. It is shown that the energy strongly depends on the derived relation between the occupancy numbers of fermions and quanta of the YM field.

The relation of the developed model to the generally accepted results in QCD is considered. In the case of the hot homogeneous equilibrium quark-gluon plasma the re-normalization of a fermion mass leads to arising the thermal mass of a quark (see Eqs.(40), (49)) which strongly depends on the matter temperature. We show that in the intermediate range of the density and temperature of the plasma, $n_0 \sim g^3 T^3$, the dependence of the quark mass on the matter temperature and coupling constant (see Eq.(50)) corresponds to the results of its calculations which have been made in the hard thermal loop approximation[29, 30] early. The hadronization as the phase transition of the first kind is considered. In the case of the hadronization into the lightest hadrons the calculated mass of such hadrons appears to be of the order of the mass of a free pion.

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Appendix A

We expand the second exponent in Eq.(20) in the series:

$$\begin{aligned}
 & \exp \left\{ -ig T_a \frac{\int_0^\varphi d\varphi' (A_\mu^a p^\mu) + \frac{i}{2} (\gamma^\mu k_\mu) (\gamma^\mu A_\mu^a)}{(pk)} \right\} \equiv \exp \left\{ -\frac{ig}{(pk)} T_a B^a(\varphi) \right\} = \\
 & 1 + \left(-\frac{ig}{(pk)} \right) (T_a B^a) + \frac{1}{2!} \left(-\frac{ig}{(pk)} \right)^2 (T_a B^a)^2 + \frac{1}{3!} \left(-\frac{ig}{(pk)} \right)^3 (T_a B^a)^3 + \frac{1}{4!} \left(-\frac{ig}{(pk)} \right)^4 (T_a B^a)^4 + \dots \\
 & B^a(\varphi) \equiv \int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu) + \frac{i}{2} (k_\mu \gamma^\mu) (A_\mu^a(\varphi) \gamma^\mu); \quad B^a(\varphi) B_a(\varphi) = \left(\int_0^\varphi d\varphi' (A_\mu^a(\varphi') p^\mu) \int_0^\varphi d\varphi'' (A_\mu^a(\varphi'') p_\mu) \right) \\
 & + i (k_\mu \gamma^\mu) (A_\mu^a(\varphi) \gamma^\mu) \left(\int_0^\varphi d\varphi' (A_\mu^a(\varphi') p_\mu) \right) \equiv (c^2 + b) \tag{A1}
 \end{aligned}$$

Taking into account of Eqs.(6)-(8), (10), (11) the even and odd terms in Eq. (A.1) can be rewritten as follows:

$$\begin{aligned}
 & \left(\exp \left\{ -ig T_a \frac{\int_0^\varphi d\varphi' (A_\mu^a p^\mu) + \frac{i}{2} (\gamma^\mu k_\mu) (\gamma^\mu A_\mu^a)}{(pk)} \right\} \right)_{even} = 1 - \frac{1}{2!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2 - 2}{4N}} \right)^2 (c^2 + b) + \\
 & \frac{1}{4!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2 - 2}{4N}} \right)^4 (c^2 + 2c^2b) - \frac{1}{6!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2 - 2}{4N}} \right)^6 (c^6 + 3bc^4) + \dots = \\
 & \cos \theta - \frac{b}{2} \left(\frac{g}{(pk)} \sqrt{\frac{N^2 - 2}{4N}} \right)^2 \frac{\sin \theta}{\theta} \tag{A2}
 \end{aligned}$$

$$\begin{aligned}
& \left(\exp \left\{ -ig T_a \frac{\int_0^\varphi d\varphi' (A_\mu^a p^\mu) + \frac{i}{2} (\gamma^\mu k_\mu) (\gamma^\mu A_\mu^a)}{(pk)} \right\} \right)_{\text{odd}} = -\frac{iqT_a B^a}{(pk)} \left(1 - \frac{1}{3!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2-2}{4N}} \right)^2 (c^2 + b) + \right. \\
& \left. \frac{1}{5!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2-2}{4N}} \right)^4 (c^2 + 2c^2 b) - \frac{1}{7!} \left(\frac{g}{(pk)} \sqrt{\frac{N^2-2}{4N}} \right)^6 (c^6 + 3bc^4) + \dots \right) = \\
& -\frac{iqT_a B^a}{(pk)} \left(\frac{\sin \theta}{\theta} + \frac{b}{2} \left(\frac{g}{(pk)} \sqrt{\frac{N^2-2}{4N}} \right)^2 \frac{\theta \cos \theta - \sin \theta}{\theta^3} \right), \tag{A3}
\end{aligned}$$

where the parameters c and b are determined in Eq.(A1).

Summing the exponents in Eq.(A.2), (A.3) we directly go to Eq.(21).

Appendix B

Due to the relativistic invariance and Eqs.(10), (11) the integral containing the first term in the square bracket in Eq.(28) is equal to zero:

$$\begin{aligned}
& \sum_{\sigma\alpha} \int \frac{d^3 p}{2p^{(0)}(2\pi)^3} \left\{ \frac{gk^\nu}{(pk)} \cdot \left[\frac{N^2-2}{4N} \frac{\tan \theta}{\theta} (A_a^\mu(\varphi) p_\mu) \right] \right\} \langle \hat{a}_{\sigma,\alpha}^\dagger(p) \hat{a}_{\sigma,\alpha}(p) + \hat{b}_{\sigma,\alpha}(p) \hat{b}_{\sigma,\alpha}^\dagger(p) \rangle = \\
& gk^\nu \frac{N^2-2}{4N} A_a^\mu(\varphi) \sum_{\sigma\alpha} \int \frac{d^3 p}{2p^{(0)}(2\pi)^3} \left\{ \frac{1}{(pk)} \cdot \left[\frac{\tan \theta}{\theta} p_\mu \right] \right\} \langle \hat{a}_{\sigma,\alpha}^\dagger(p) \hat{a}_{\sigma,\alpha}(p) + \hat{b}_{\sigma,\alpha}(p) \hat{b}_{\sigma,\alpha}^\dagger(p) \rangle = \\
& k^\nu f_1(\varphi) (A_a^\mu(\varphi) k_\mu) = 0, \tag{B1}
\end{aligned}$$

where $f_1(\varphi)$ is some scalar function.

Because of the relativistic invariance, the fact that J_a^ν is a $SU(N)$ vector, and Eqs.(10), (11) the integral containing the second term in the square bracket in Eq.(28) is also equal to zero:

$$\begin{aligned}
& \sum_{\sigma\alpha} \int \frac{d^3 p}{2p^{(0)}(2\pi)^3} \left\{ \frac{gk^\nu}{(pk)} \cdot \left[g^2 \left(\frac{N^2-2}{4N} \right)^2 \frac{\theta \tan^2 \theta + \theta - \tan \theta}{(pk)^2 \theta^3} \int_0^\varphi d\varphi' (A_a^\mu(\varphi') p_\mu) \right. \right. \\
& \left. \left. \left((A_b^\mu(\varphi) p_\mu) \int_0^\varphi d\varphi' (A_\mu^b(\varphi') p^\mu) \right) \right] \right\} \langle \hat{a}_{\sigma,\alpha}^\dagger(p) \hat{a}_{\sigma,\alpha}(p) + \hat{b}_{\sigma,\alpha}(p) \hat{b}_{\sigma,\alpha}^\dagger(p) \rangle = \\
& g^3 k^\nu \left(\frac{N^2-2}{4N} \right)^2 A_b^\rho(\varphi) \int_0^\varphi d\varphi' A_a^\mu(\varphi') \int_0^\varphi d\varphi'' A_\lambda^b(\varphi'') \sum_{\sigma\alpha} \int \frac{d^3 p}{2p^{(0)}(2\pi)^3} \left\{ \frac{p_\rho p^\lambda p_\mu}{(pk)} \cdot \frac{\theta \tan^2 \theta + \theta - \tan \theta}{(pk)^2 \theta^3} \right\} \\
& \langle \hat{a}_{\sigma,\alpha}^\dagger(p) \hat{a}_{\sigma,\alpha}(p) + \hat{b}_{\sigma,\alpha}(p) \hat{b}_{\sigma,\alpha}^\dagger(p) \rangle = \\
& k^\nu A_b^\rho(\varphi) \int_0^\varphi d\varphi' A_a^\mu(\varphi') \int_0^\varphi d\varphi'' A_\lambda^b(\varphi'') (k^\lambda k_\rho k_\mu f_2(\varphi) + \mathcal{G}_{\rho\mu} k^\lambda f_3(\varphi) + \mathcal{G}_\mu^\lambda k_\rho f_4(\varphi) + \mathcal{G}_\rho^\lambda k_\mu f_5(\varphi)) = 0 \tag{B2}
\end{aligned}$$

where $f_i(\varphi)$ are some scalar functions.